国際シンポジウム第１日目　講演４

Fluctuation, Nonlinearity and for Human Beings

Takeyuki Hida*

§ 1. Fluctuation

The most important and basic mathematical model of fluctuation is a white noise, which is practically obtained by taking the time-derivative of a Brownian motion B(t):

$$\dot{B}(t) = dB(t)/dt.$$

The collection of $\dot{B}(t)$'s forms a system of idealized elementary random variables (abbr. i.e.r.v.). This means that it is an independent system and each member is atomic (may be called to be elementary or primitive).

A white noise is, in addition, such that the probability distribution is Gaussian and the member is an infinitesimal random variable.

The significance of a white noise is not only because of this reason, but also comes from the fact that many of its properties are inherited from those of Brownian motion, which is the most important example of a stochastic process.

Our interest is to investigate random phenomena which are expressed as functionals of white noise. This suggests that we should first clarify the probabilistic properties of white noise. For this purpose we shall have a short review of a Brownian motion.

§ 2. Characteristic properties of a Brownian motion

(1) Probability distribution of a Brownian motion is Gaussian.

There is no need to emphasize the importance of Gaussian distribution. Still, we should like to say a few words on this distribution.

It has maximum entropy under the restriction that the variance (power) is limited. This means that Gaussian variables are most suitable for the information source, while they are worst if they occur as disturbance of a communication system.

The next remark is that a Gaussian system of random variables enjoys a linear structure. The joint probability distribution is uniquely determined by the mean vector and the covariance matrix. So far as linear operations acting on the system are concerned, it remains to be Gaussian for over.

* 名城大学理工学部長

One more remarkable fact is that the white noise measure which is the probability distribution of white noise is closely related to the ideal measure on the Hilbert space $L^2[0, 1]$, on which the classical functional analysis has been carried out successfully.

(2) Sample function properties.

To denote a sample function of a Brownian motion a probability parameter ω is introduced. Now we claim the uniform continuity of $B(t, \omega)$. Let us assume t and t+h run through a finite time interval. Then, if h is small (depending only on ω), we have

$$|B(t+h, \omega)-B(t, \omega)| \text{ is of order } \sqrt{|h| \log(1/|h|)}$$

uniformly in t.

Again we assume that t runs through a finite interval. Then, the second order variation is positive and finite. For instance,

$$\Sigma_i(\Delta_i B)^2 \longrightarrow 1, t \in [0, 1].$$

holds.

(3)Geometric properties of two-dimensional Brownian motion.

Hausdorff dimension or fractal dimension can be considered.

Take a function

$$\Phi(\rho) = \rho^2 \log\log(1/\rho).$$

The induced measure of a trajectory for the time interval [0, t] is shown to be proportional to t.

A Brownian particle cruises on the plane for occupying the territory as wide as possible. This property is illustrated in many ways.

Also, it is noted that the second order Hausdorff measure of the path is 0.

(4) Approximations.

Brownian motion creates infinite dimensional analysis, although for the actual computations we often require some finite dimensional approximation.

Here are some typical methods of approximation. Each method is, of course, used depending on the purpose.

i) Random walk.

An ordinary symmetric random walk is modified in such a way that each step is performed during 1/n time unit and scale is multiplied by $1/\sqrt{n}$. Then, roughly speaking, the modified random walk approaches to Brownian motion, as $n \longrightarrow \infty$.

ii) Successive interporation.

The Lévy's construction (see(4) Chapt.1.) of a Brownian motion using the interporation technique is suggestive for our white noise analysis, which may be viewed as a typical infinite dimensional analysis. The main reason to be suggestive is that, through the construction of a Brownian motion one can consider the particular way how the infinite dimensional objects are successively approached by a sequence of finite dimensional ones. In fact, every axis sprits into many directions uniformly in the axis. This idea reflects on some subgroup of the infinite dimensional rotation group.

iii) Band limited white noise.

Take a white noise instead of a Brownian motion. Since it has stationary independent increments, the white noise is a stationary (generalized) stochastic process, and the spectral density function is a constant ; it is a process with flat spectrum. Let the spectrum be limited to a finite interval, say [-W, W], by cutting higher frequencies. Still, we are given a stationary Gaussian process and now its sample functions are analytic, quite unlike those of the original white noise, the sample functions of which are generalized functions. However, in a sense, the band limited one well approximates a white noise. To get a Brownian motion, just take an integral over the interval [0, t].

§ 3. White noise functionals

We are interested in random phenomena that are expressed mathematicaly as functionals of white noise. Namely, we wish to analyze a functional that is expressed in the form

$$\phi(\dot{B}) = \phi(\dot{B}(t), t \in [0, 1]).$$

Note that $\{\dot{B}(t)\}$ is taken to be the system of variables of ϕ.

With this choice of variables, it is quite natural to define the polynomials in $\dot{B}(t)$'s. Unfortunately, the $\dot{B}(t)$, more precisely $\dot{B}(t, \omega)$, is a generalized function of t, and so the usual method in analysis does not permit to form polynomials in the $\dot{B}(t)$. Hence, some modificationis are requested. One might thing of smearing the $\dot{B}(t)$. But we are not willing to do so. Because t stands for the time, so the t should survive for our favourite causal calculus, where the time development is always taken into account in terms of t.

On the other hand, we see that probability distribution of $\{\dot{B}(t)\}$ is a standard Gaussian measure μ defined on a space E^* of generalized functions on a Euclidean space. We certainly recognize the importance of the measure μ.

In favour of the above considerations, we are now in a position to propose the background of our analysis.

Making a long story short, we have come to a Gel'fand triple starting from the Hilbert space $(L^2) = L^2(E^*, \mu)$:

$$(S) \subset (L^2) \subset (S)^*,$$

where (S) is the space of test functionals and (S)* is that of generalized (white noise) functionals.

This triple is an infinite dimensional analogue of that in the case of the Schwartz distribution:

$$S \subset L^2(R) \subset S',$$

where S' is the space of tempered distributions.

Now we are ready to discuss white noise analysis on our favorite space (S)*. Some recent developments of the white noise analysis have allowed to make a first breakthrough in some important problems in science and technology. In what follows, having prepared some more background, we proceed to discuss significant questions in various applications which could be approached by our white noise analysis with special emphasis on the aspects of information theory.

Since the system of variables of the functional in question is fixed, it is quite reasonable to define a differential operator in the variable $\dot{B}(t)$:

$$\partial_t = \partial / \partial \dot{B}(t).$$

It can be proved that the domain of ∂_t is rich enough, including the test function space (S), and the ∂_t is an annihilation operator.

Its adjoint ∂_t^* plays a role of a creation. It defines a stochastic integral called the Hitsuda-Skorohod integral.

The annihilation and the creation operators satisfy the commutation relations:

$$[\partial_t, \partial_s^*] = \delta(t-s).$$

Since we are dealing with an infinite dimensional analysis, various kinds of Laplacian could be introduced : Δ_∞(Laplace-Beltrami operator), Δ_V(Volterra Laplacian), Δ_L(Lévy Laplacian), and so forth. They can play their own roles acting on different domains.

There is introduced the infinite dimensional rotation group O_∞, since the white noise measure is like a uniform measure on the infinite-dimensional sphere. Hence we can speak of the harmonic analysis arising from the rotation group, as we have briefly mentioned before. This topic, however, we do not go into details.

Thus, White Noise Analysis has been established.

§ 4. Applications 1

This and the next sections shall be devoted to the discussions on the Input-Output Dynamics by using our white noise analysis.

First, consider the case where a white noise is taken to be the input to the system.

Let U_t be the unitary operator acting on (L^2) corresponding to the time-shift, which comes from the transformation of the white noise:

$$\dot{B}(\cdot) \longrightarrow \dot{B}(\cdot + t).$$

The system $\{U_t\ ;\ t \text{ real}\}$ forms a continuous one-parameter group of unitary operators acting on (L^2):

$$U_t U_s = U_{t+s},$$
$$U_t \longrightarrow I \text{ (identitry), as } t \longrightarrow 0.$$

We can therefore appeal to Stone's theorem to establish the spectral representation of U_t, and hence spreak of the cyclic subspace, the spectral multiplicity, spectral measure and so forth.

It is known that there is a direct sum decomposition of the Hilbert space (L^2) called the Fock space:

$$(L^2) = \oplus_n H_n,$$

where H_n is the space of homogeneous chaos of order n formed by Hermite polynomials in $\dot{B}(t)$'s of degree n. The H_n is also called the space of multiple Wiener integrals of degree n.

It is easy to prove that the $\{U_t\}$ has simple multiplicity on H_1. White, it has countably infinite multiplicity on H_n, $n>1$, so that it admits much finer direct sum decomposition into infinitely many cyclic subspaces, where $\{U_t\}$ has unit multiplicity.

By definition, the structure of the subspace having unit multiplicity is determined by the associated spectral measure. In the present case of white noise, all the spectral measure are absolutely continuous with respect to the Lebesgue measure.

The properties mentioned above play dominant role in various applications which will be discussed below. Also, we are naturally led to the so-called Wiener expansion of nonlinear (often generalized) white noise functionals (see(2), § 5.5).

Case 1. Linear system. Analysis on H_1.

Remind that the action of U_t, and hence the spectral density function, completely determines the structure of the system.

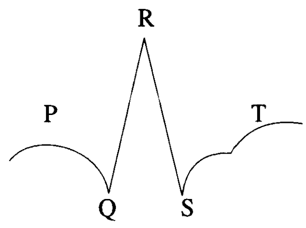

Fig. 1

Example 1. An LCR network, in particular all-pass network.

Given a white noise input. Then, we are given a stationary Gaussian process through an LCR network. The characteristics of the network will determines the spectrum an hence the covariance function of the process. With this knowledge we can discuss filtering problems, prediction of the future values of the process and other topics which are related to the causal calculus of white noise functionals.

It is noted that these problems have close connection with the optimal factorization of the covariance functions. (Or more generally, that of positive definite functions).

Example 2. An example of a discrete time stationary stochastic process.

Irregular heartbeat produces a discrete time stochastic process (time series). The PQRS data (Fig.1) are viewed as samples of a discrete time stationary process (Sawada). In particular, we focus our attention to their time intervals of peaks. The spectrum of the process is used for diagnosis of the heart disease.

Case 2. Nonlinear system.

Example 3. Ken-ichi Naka and Vanita Bhanot. "Identification of catfish retina."

The input is taken to be white noise and through the retina the output is given. It is a stationary stochastic process expressed as nonlinear functional of the input white noise.

Naka and Bhanot have appealed to the Wiener expansion of the output. Actually, the basic recipe is the combination of the Fock space and the action of the unitary group $\{U_t\}$ on each cyclic subspace. The choice of finite number of significant cyclic subspaces is the crucial problem and the decision would come from their biological insight. See (5).

The technique developed in this example can be applied to many other problems and the result mentioned above would be a remarkable step towards the case of human beings.

§5. Applications 2

We then discuss input-output dynamics, where the input is unknown. There are several interesting cases, where white noise analysis can be efficiently applied.

Case 1. Output signal is Gaussian.

Denote the output by $X(t)$. Assume that it is observed to be Gaussian. Then, we can apply the representation theory of the Gaussian process in terms of white noise. Actually, $X(t)$ is expressed in the form

$$X(t) = \int^{t} F(t, u)\ \dot{B}(u)\, du,$$

where we tacitly assume that $X(t)$ is centered, that is the mean $E(X(t))$ has already been subtracted off.

The kernel function F is obtained by a factorization of the covariance function of $X(t)$. Among various candidates of F there exists one optimal kernel. The representation with such a kernel is called canonical. Once the canonical representation is given, we shall see all the stochastic properties of $X(t)$.

How to get the best predictor is an easy consequence of this result.

Example 4. (Yonezawa 1996) Fluctuation of human body.

We are interested in the fluctuation of the center of gravitation of human body. It is expressed as the two dimensional Gaussian process and hence the theory of representation stated above should be generalized slightly. The kernels in the representation can well characterize the fluctuation in question.

Case 2. Input is unknown and output is not Gaussian.

Consider such cases where we just observe the output, which is no more Gaussian. We wonder what is the input. In reality, in many cases we many assume that the input is Gaussian. Such a guess comes, depending on the situation, from the characteristic properties of a Brownian motion or a white noise: those properties we have reviewed in §2. Suppose the guess is the case, a whitening can be applied (by a similar method to form the canonical representation as in Case 1) and we may assume that white noise is the information source.

For the purpose of justifying the above understanding, we are going to propose several methods of practice.

i) Regularity of sample function (observed data).

(a) Suppose the output process $X(t)$ is a stationary stochastic process. Such an assumption is reasonable, since the output should share the time shift with the input. Then, if the input is Gaussian,

$$dX(t) \sim dB(t)$$

must hold, ignoring higher order term. (See the Wiener expansion.)
Thus, we know that the local regularity of sample paths is somewhat limited.

(b) One of the most interesting global properties of a sample function is perhaps self-similarity. We can observe such a property in the astrophysical data by Oda (6). Hopefully, we may assume that it is a stable stochastic process. For further investigation, we wish to determine the exponet α of the stable process. To estimate α we need to compute the characteristic functional, which is obtained from a sample function by using the ergodic property.

ii) There is the Potthoff-Streit characterization of white noise functionals ((3), § 4.C).

The criterion is given in terms of the so-called S-transform, so that it is requested to apply the transformation to the output functional, and we can see if the input is Gaussian or not.

iii) Innovation approach to random fields.

For a stochastic process X(t) is the stochastic infinitesimal equation proposed by P. Lévy:

$$\delta X(t) = \Phi(X(s), s \leqq t\, ; Y(t), t, dt),$$

where Φ is a sure function and $\{Y(t)\}$ is the innovation. The equation above has only a formal significance, however it suggests us how to find the probabilistic structure of the X(t).

We know some particular and significant cases where the innovation is obtained. In those cases, we are ready to carry on white noise analysis. This idea can be extended to the case of a random field X(C) depending on a closed curve or surface C that runs through a space-time region. A generalization of the stochastic infinitesimal equation may be proposed to be

$$\delta X(C) = \Phi(X(C'), C' < C\, ; Y(s), s \in C, C, \delta C),$$

where $C' < C$ means that the region (C') enclosed by C' is a subset of the region (C), and where $\{Y(s)\}$ is the innovation.

The approach in this line by forming X(C) from actual random phenomena would be powerful and be suggested by many reasons.

Example 5. (Si Si (7)) Having established a generalization of the infinitesimal equation to the case of a random field, the importance of the innovation approach is realized.

Let X(C) be a random field of a homogeneous chaos, say of degree n, where C denotes a smooth manifold without boundary. Then, form the variations $\delta X(C)$ and apply some nonlinear operations to obtain the innovation. The innovation is used to investigate the given field by applying the white noise analysis.

Remark. For a random field X(C) we see that by infinitesimal deformation δC of C we are given extremely rich information from $\delta X(C)$ compared to the case of dX(t) for X(t). It is therefore highly recommended, in the actual information processing, to have observation not as a time series X(t) depending on t, but as a random field X(C) indexed by a space-time figure C. As for this idea, Professor M. Oda has kindly suggested to the author to read the book (1), where a similar idea to take random fields is illustrated. The readers are recommended to see the book to obtain further understanding in line with white noise analysis.

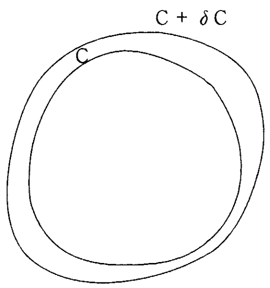

Fig. 2

Acknowledgements. The author is grateful to Professor Minoru Oda, Rektor of Tokyo University of Information Sciences, who has given me the opportunity to deliver a lecture on the occasion of the opening of High Tech Research Center. Most of my talk is summarized in these notes. They have also been prepared in line with the joint research program by this High Tech Research Center and the Academic Frontier project at Meijo University. The author is grateful to both projects for the support.

References

(1) R. Hanbury Brown, The Intensity Interferometer. Its Applications to Astronomy. Taylor & Francis LTD. 1974.

(2) T. Hida, Brownian motion. Springer-Verlag, 1980.

(3) T.Hida, H.-H.Kuo. J.Potthoff & L. Streit, White Noise. An Infinite Dimensional Calculus. Kluwer Academic. Pub. 1993.

(4) P. Lévy, Processus Stochastiques et Mouvement Brownien. Gauthier-Villars. 1948.

(5) K.-I.Naka and V. Bhanot, White-noise analysis in retinal physiology. Advanced Mathematical Approach to Biology. World Sci. 1997, 109-267.

(6) M.Oda, Fluctuation in astrophysical phenomena. Mathematical Approach to Fluctuations. World Sci. 1994, 115-137.

(7) Si Si, Innovation of some random fields. Proc. of the Conference on Probability Theory and its Applications. Taejon, 1998.